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NOTE

Numerical Solution to the Heat-Transfer Equations with Combined Conduction and Radiation[1]

When the differential equations governing combined conduction and radiation steady-state heat transfer are written in finite-difference form, a system of algebraic equations for the temperatures, T_i, of the form

$$\sum_{j=1}^{N} A_{ij}T_j + \sum_{j=1}^{N} B_{ij}T_j^4 = C_i \qquad i = 1, 2, \ldots, N \tag{1}$$

results, where A_{ij} and B_{ij} are $N \times N$ diagonally dominant Stieltjes matrices and T_i and C_i are column vectors of dimension N. T_i^4 is a column vector formed by raising each term of the T_i vector to the fourth power. This report presents an efficient method of solving Eq. (1).

The form of B_{ij} permits a linearization of the radiation term according to the scheme ($B_{ij} < 0, i \neq j$):

$$\tilde{R}_{ij} = \begin{cases} B_{ij}(\tilde{T}_i + \tilde{T}_j)(\tilde{T}_i^2 + \tilde{T}_j^2), & i \neq j, \\ -\sum_{k=1}^{N} \tilde{R}_{ik} + 4\left(B_{ii} + \sum_{k \neq i}^{N} B_{ik}\right)\tilde{T}_i^3, & i = j. \end{cases}$$

so that the matrix $\{\tilde{R}_{ij}\}$ resulting from the linearization is also of the diagonally dominant Stieltjes type. Under the linearization the vector on the right-hand side must be modified to be of the form

$$\tilde{C}_i = C_i + 3\left(B_{ii} + \sum_{k \neq i}^{N} B_{ik}\right)\tilde{T}_i^4.$$

If the linearized equations are solved for T_i and $\{\tilde{R}_{ij}\}$ is recalculated, an iteration process can be devised that closely resembles the Newton–Raphson method. The computation time of such a scheme exceeds that of the method now proposed.

Equation (1) can be written in the form

$$\sum_{j=1}^{N} (A_{ij} + \tilde{R}_{ij} + D_{ij})\,T_j = \tilde{C}_i + \sum_{j=1}^{N} \tilde{R}_{ij}T_j - \sum_{j=1}^{N} B_{ij}T_j^4 + \sum_{j=1}^{N} D_{ij}T_j,$$

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where the diagonal matrix $\{D_{ij}\}$ has been added to both sides of the equation. If the inverse of the matrix $\{A_{ij} + \tilde{R}_{ij} + D_{ij}\}$ is denoted $\{E_{ij}\}$, then the proposed iteration scheme is

$$T_i^{(n+1)} = \sum_j E_{ij} \left[\tilde{C}_j + \sum_{k=1}^{N} \tilde{R}_{jk} T_k - \sum_{k=1}^{N} B_{jk} T_k^4 + D_{jj} T_j \right]^{(n)},$$

where the bracketed term is evaluated using $T_i^{(n)}$. Convergence of this scheme can be studied by linearizing the radiation term by the true solution $T_i^{(\infty)}$. Convergence depends on ρ, the spectral radius of the iteration matrix $\{Z_{ik}\}$, where

$$Z_{ik} = \sum_j E_{ij} [\tilde{R}_{jk} + D_{jk} - R_{jk}^{(\infty)}].$$

The separation

$$\{A_{ij} + \tilde{R}_{ij} + D_{ij}\} - \{D_{ij} + \tilde{R}_{ij} - R_{ij}^{(\infty)}\}$$

is a regular splitting [1] of the matrix

$$\{A_{ij} + R_{ij}^{(\infty)}\}$$

provided

$$\{D_{ij} + \tilde{R}_{ij} - R_{ij}^{(\infty)}\} > 0. \tag{3}$$

Therefore, if inequality (3) is satisfied, the conditions for Theorem (2.2) of Varga [1] are satisfied and $\rho(z) < 1$. Inequality (3) is then a sufficient condition for the convergence of the iteration process and can be used to calculate the required D_{ij} from the estimated value of $R_{ij}^{(\infty)}$. In particular, if we select $D_{ij} = 0$ for $i \neq j$, then we must have

$$\tilde{T}_i < T_i^{(\infty)} \tag{4}$$

and

$$D_{ii} > R_{ii}^{(\infty)} - \tilde{R}_{ii}. \tag{5}$$

In practice, we must estimate $T_i^{(\infty)}$; let this estimate be T_i'. Then, since we must set

$$D_{ii} = R_{ii}' - \tilde{R}_{ii}, \tag{6}$$

T_i' must be selected equal to or greater than $T_i^{(\infty)}$ for (3) to hold. Although no mathematical proof can be given as yet, provided T_i' is greater than $T_i^{(\infty)}$, (4) can be violated and convergence obtained if $\tilde{T}_i$ is less than T_i' and D_{ii} is determined from (6). In fact, $\tilde{T}_i$ should be chosen as close to the true solution as possible.

Applications

The proposed scheme, Eq. (2), was applied to a variety of heat-transfer problems with N ranging from 8 to 64. Details of some of the problems are given in [3]. A comparison of solution times on an IBM 7094 digital computer are shown in reduced form on Table I. The advantages of the present method are apparent.

TABLE I

Solution Times for Iteration Schemes[a]

Iteration Scheme	T^4-dominated	Problem T-dominated	Equal-T, T^4
Proposed Scheme ($\hat{T} < 1.3T^{(\infty)}$)	1.8	0.8	1.3
Successive Over-Relaxation	1.3	1.3	1.8
Maximum Rate of Descent	2.7	2.7	3.2
Conjugate Gradient	1.6	1.2	1.4

[a] M secs/(equations)$^{1.8}$ (initial error, percent)$^{0.4}$.

Of greater significance is the fact that in some typical spacecraft problems, where $\{A_{ij}\}$ is singular and $\{B_{ij}\}$ is nearly so, the proposed method proved at least ten times faster than any of the others. For many applications the general applicability of the proposed method would indicate its choice.

The proposed technique can be readily adapted the other nonlinearities.

Acknowledgments

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References

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